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The Clifford algebra of nonrelativistic phase space and the concept of mass

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Abstract

Prompted by a recent demonstration that the structure of a single quark–lepton generation may be understood via a Dirac-like linearization of the form $\mathbf{p}^2+\mathbf{x}^2$, we analyse the corresponding Clifford algebra in some detail. After classifying all elements of this algebra according to their $U(1) \otimes SU(3)$ and $SU(2)$ transformation properties, we identify the element which might be associated with the concept of lepton mass. This element is then transformed into a corresponding element for a single coloured quark. It is shown that—although none of the three thus obtained individual quark mass elements is rotationally invariant—the rotational invariance of the quark mass term is restored when the sum over quark colours is performed.

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1. Introduction

The present paper develops somewhat further the approach proposed in [1–4]. The ideas and heuristic arguments which provide a conceptual justification for the whole approach were originally presented in [4]. They stemmed from (1) dissatisfaction with the way quark mass is introduced and used in contemporary elementary particle physics, and (2) a wish to introduce more symmetry between position and momentum[1]. In [4] it was argued that instead of identifying the arena of nonrelativistic physics with the observable three-dimensional space,

[1] While these two arguments are as valid as ever, one may now say with a hindsight that the particular way in which symmetry between position and momentum was enforced in [4] suffered from an erroneous use of $U(1) \otimes SU(3)$ as a means of effecting the anticipated lepton–quark interchange. Such a use of $U(1) \otimes SU(3)$ was forced by the requirement imposed in [4] that the position–momentum Poisson bracket be invariant. It appears now, as found in [3], that the corresponding invariance of the position–momentum commutation relations has to be somewhat relaxed and admit arbitrary (+ or −) signs in front of the imaginary unit, independently for each of the three directions of our macroscopic 3D world. The lepton–quark interchange is then affected by transformations outside of $U(1) \otimes SU(3)$.

one should adopt the description given by the nonrelativistic Hamiltonian formalism, in which momentum and position coordinates are treated as independent variables. In this language, the relevant arena appears to be that of phase space. As a result, the question of a possible symmetry between the momentum and position coordinates (advocated long ago by Born [5]) may be then formulated in a more natural way. This redefinition of what constitutes an arena admits a generalization of the way in which 'canonical momenta' and 'canonical positions' are identified with physical momenta and physical positions. Thus, it appears that instead of just one way suggested by the 3D formalism, one can perform such an identification in four ways (including the old 3D one). Reference [4] also put forward a conjecture concerning the generalization of the standard concept of mass. It consisted in associating the concept of mass not only with the physical momentum (as in standard relations between energy, mass and momentum), but, more generally, with the very four types of 'canonical momenta'. It was then further suggested that the additional three ways of assigning the concept of mass are related to the existence of quarks, and that it is the ensuing lack of rotational invariance which is connected with both quark unobservability and the conceptual problems related to the way quark mass is introduced in contemporary physics.

In [1] the above idea was developed further by admitting non-commuting positions and momenta. Subsequent Dirac-like linearization of the basic phase-space invariant $R^z \equiv \mathbf{p}^2 + \mathbf{x}^2$ has led to the appearance of a corresponding matrix operator R in Clifford algebra[2], and to the proposal of its identification (up to a factor) with the hypercharge operator. Similarly, the operator $R^{\mathrm{tot}} = R^z + R$, with the lowest 'vacuum' eigenvalue adopted for R^z, was conjectured to be identical (up to a factor) to the charge operator Q. The resulting set of the eigenvalues of Q corresponds precisely to the set of quark and lepton charges of a single standard model (SM) generation. At the same time, the study of the relevant Clifford algebra provided a raison d'être for the appearance of the symmetry group $U(1) \otimes SU(3)$ combined with $SU(2)$, a conjectured precursor of the SM gauge group. Furthermore, it was shown [2, 3] that there is a one-to-one correspondence between the way in which charge eigenvalues emerge in the proposed scheme and in the Harari–Shupe rishon model [6]. Finally, a phase-space-based interpretation of the connection between leptons and quarks was proposed, with quarks being related to leptons via genuine rotations in phase space and weak isospin related to reflections in phase space [2, 3].

The success of our approach has its origin in the application of the concept of Clifford algebra to nonrelativistic phase space. Although many authors stressed in various contexts the importance of Clifford algebras in physics (see e.g. [7, 8]), applications of this concept to phase space are fairly rare (see [9]). It is therefore appropriate to study the structure of the Clifford algebra of nonrelativistic phase space in some detail. This is all the more important because our phase space approach necessarily introduces a fundamental constant of dimension (momentum/position) which—when the Planck constant is taken into account—should set a natural mass scale. Thus, the hope is that our Clifford algebra contains not only the generators of the relevant symmetries and the algebraic counterparts of positions and momenta, but will also provide us with some ideas concerning the algebraic approach to the concept of mass. With this in mind, we shall first classify all elements of the Clifford algebra in question according to their $U(1) \otimes SU(3)$ and $SU(2)$ properties. Then, we will linearize the nonrelativistic relation between kinetic energy, mass and momentum and identify those elements of Clifford algebra which may be associated with the concept of lepton mass. Finally, using the lepton–quark transformations introduced in [3], we will transform these elements from the lepton to the

[2] The superscript σ, used in [1–3] to distinguish between the matrix and phase-space representations, is suppressed throughout the present paper.

quark sector. In this way, we obtain three elements of the Clifford algebra, each corresponding to the mass of an individual (coloured) quark. We will then show that, although each of these elements, when taken separately, is not invariant under ordinary rotation, the three elements add up to give a rotationally-invariant total quark mass term.

We treat our Clifford algebra of nonrelativistic phase space as a laboratory and testing ground. The real world is clearly much more complex than what this algebra suggests. Yet, it should be pointed out that our scheme treats the ordinary 3D rotation in the only natural way, i.e. as *a pair of identical (same size and sense) rotations in the momentum and position subspaces of the phase space*. Consequently, all our conclusions regarding the behaviour under ordinary 3D rotations should be valid in general.

2. Basic definitions, hypercharge and isospin

We shall denote the basic elements of the relevant Clifford algebra by A_k and B_l, with $A_k(B_l)$ associated with nonrelativistic momentum (position), and use the following explicit representation:

$$\begin{aligned} A_k &= \sigma_k \otimes \sigma_0 \otimes \sigma_1, \\ B_k &= \sigma_0 \otimes \sigma_k \otimes \sigma_2, \\ B &\equiv \mathrm{i}A_1A_2A_3B_1B_2B_3 = \sigma_0 \otimes \sigma_0 \otimes \sigma_3. \end{aligned} \tag{1}$$

In order to analyse our Clifford algebra in terms of its $U(1)\otimes SU(3)$ properties, it is appropriate to introduce combinations analogous to the standard annihilation and creation operators a_k and $a_k^\dagger$, i.e.:

$$C_k = \tfrac{1}{\sqrt{2}}(B_k + \mathrm{i}A_k), \qquad C_k^\dagger = \tfrac{1}{\sqrt{2}}(B_k - \mathrm{i}A_k). \tag{2}$$

The anticommutation relations satisfied by elements A_k, B_l, B translate then into

$$\begin{aligned} &\{B, C_k\} = \left\{B, C_k^\dagger\right\} = \{C_k, C_l\} = \left\{C_k^\dagger, C_l^\dagger\right\} = 0, \\ &\left\{C_k, C_l^\dagger\right\} = 2\delta_{kl}. \end{aligned} \tag{3}$$

The total of 64 elements of Clifford algebra may be grouped into four sets of 16 elements each. The first two sets are composed of linear combinations of the products of an even number of A_k, B_l, while the latter two sets are built of linear combinations of the products of an odd number of A_k, B_l. Before we proceed with the full presentation of all elements of the Clifford algebra, we need to introduce two important even elements: the hypercharge Y and the third component of the weak isospin I_3.

2.1. Hypercharge

In line with [1, 2], the hypercharge is defined as

$$Y = \tfrac{1}{3}\mathcal{Y}, \tag{4}$$

where (summation convention over repeated indices implied)

$$\mathcal{Y} = \sum_k \mathcal{Y}_k = -\frac{\mathrm{i}}{2}[A_k, B_k]B = -\frac{1}{2}\left[C_k, C_k^\dagger\right]B. \tag{5}$$

The 4×4 analogues of $\mathcal{Y}$ and $\mathcal{Y}_k$ will be denoted by y and y_k:

$$\mathcal{Y} = y \otimes \sigma_0 = \sum_k \mathcal{Y}_k = \sum_{k=1}^{3} y_k \otimes \sigma_0 = \sigma_k \otimes \sigma_k \otimes \sigma_0. \tag{6}$$

One finds that y and y_k's satisfy the following equations:

$$\begin{aligned} & y_1^2 = y_2^2 = y_3^2 = +1, \\ & y_i y_j = -y_k \qquad (i \neq j \neq k \neq i), \\ & y_1 y_2 + y_2 y_3 + y_3 y_1 = -y, \\ & y_1 y_2 y_3 = -1, \\ & y^2 + 2y - 3 = 0. \end{aligned} \tag{7}$$

Matrices y_k, y commute among themselves: $[y_k, y_l] = [y_k, y] = 0$ for any k, l and, consequently, they may be simultaneously diagonalized. Analogous statements hold for $\mathcal{Y}$ and $\mathcal{Y}_k$. From the last equation in (7) it follows that the eigenvalues of y are $+1$ (which is triple degenerate) and -3. The three ways of building the eigenvalue $y = +1$ out of the eigenvalues ± 1 of y_1, y_2 and y_3 are identified with the colour degree of freedom. The corresponding eigenvalues of Y are $+\frac{1}{3}$ and -1, as appropriate for the description of coloured quarks and leptons. For more details, see [1–3].

2.2. Weak isospin

The weak isospin I_3 is related to the seventh anticommuting element of the algebra:

$$I_3 = \tfrac{1}{2} B. \tag{8}$$

Furthermore (using the convention that underlined repeated indices are not summed over), we define the following products of B, $\mathcal{Y}$ and $\mathcal{Y}_k$:

$$R_k = \mathcal{Y}_k B = -\tfrac{1}{2}\left[C_{\underline{k}}, C_{\underline{k}}^\dagger\right], \qquad R = \mathcal{Y} B = -\tfrac{1}{2}\left[C_k, C_k^\dagger\right]. \tag{9}$$

As is easily checked, all elements introduced so far, i.e. $\mathcal{Y}$, $\mathcal{Y}_k$, B, R and R_l, commute with each other. Following the proposal of [1], the charge operator is then naturally obtained from the linearization of R^z as

$$Q = I_3 + \frac{Y}{2}. \tag{10}$$

2.3. Projection operators

In the following, we will need projection operators for the subspaces of definite I_3 and Y. For the $I_3 = \pm\frac{1}{2}$ isospin subspaces, they are given by

$$I_{\pm\frac{1}{2}} = \tfrac{1}{2} \pm I_3. \tag{11}$$

For the subspaces of hypercharge $Y = -1, +\frac{1}{3}$ the projection operators are

$$Y_{-1} = \frac{1 - \mathcal{Y}}{4}, \qquad Y_{+\frac{1}{3}} = \frac{3 + \mathcal{Y}}{4}. \tag{12}$$

As discussed in [1–3], the colour subspace $\#k$ is characterized by the set of eigenvalues ($y_k = -1$, $y_{l_i} = +1$ for $l_{1,2} \neq k$). The relevant projection operator is then

$$Y_{+\frac{1}{3},k} = \frac{1}{4}(1 + \mathcal{Y} - 2\mathcal{Y}_k) = \frac{3 + \mathcal{Y}}{4} \cdot \frac{1 - \mathcal{Y}_k}{4}, \tag{13}$$

and it obviously satisfies

$$\sum_k Y_{+\frac{1}{3},k} = Y_{+\frac{1}{3}}. \tag{14}$$

In the subsequent sections, the products of projection operators in the $I_3 = \pm 1/2$ and $Y = -1, +1/3$ subspaces will often occur. Consequently, it is appropriate to introduce the following compact notation:

$$Y_{-1}^{\pm} = I_{\pm\frac{1}{2}} Y_{-1}, \qquad Y_{+\frac{1}{3}}^{\pm} = I_{\pm\frac{1}{2}} Y_{+\frac{1}{3}}, \qquad Y_{+\frac{1}{3},k}^{\pm} = I_{\pm\frac{1}{2}} Y_{+\frac{1}{3},k}. \tag{15}$$

3. Even elements of Clifford algebra

The 15 generators of $SU(4)$ are represented via the commutators among A_k, B_l. In terms of $C_k, C_l^\dagger$, the relevant shift operators are expressed as follows:

$$\begin{aligned} H_{kl} &= -\tfrac{1}{4}\left[C_k, C_l^\dagger\right] = (H_{lk})^\dagger, \\ H_{m0} &= -\tfrac{1}{8}\epsilon_{mkl}[C_k, C_l] = -\tfrac{1}{4}\epsilon_{mkl}C_kC_l, \\ H_{0m} &= +\tfrac{1}{8}\epsilon_{mkl}\left[C_k^\dagger, C_l^\dagger\right] = +\tfrac{1}{4}\epsilon_{mkl}C_k^\dagger C_l^\dagger. \end{aligned} \tag{16}$$

The set of H_{kl}'s contains the $U(1)$ generator H_{kk} and the eight traceless $SU(3)$ shift operators $H_{kl} - \frac{1}{3}\delta_{kl}H_{mm}$. Elements H_{m0} and H_{0m} constitute the 'genuine' $SU(4)$ shift operators.

Since elements H_{kl}, H_{m0} and H_{0m} commute with I_3, it is natural to introduce their projections onto the $I_3 = \pm\frac{1}{2}$ subspaces:

$$\begin{aligned} H_{nk}^\pm &= H_{nk}I_{\pm\frac{1}{2}}, & H_{n0}^\pm &= H_{n0}I_{\pm\frac{1}{2}}, \\ H_{0n}^\pm &= H_{0n}I_{\pm\frac{1}{2}}, & I_{\pm\frac{1}{2}} &= 1\cdot I_{\pm\frac{1}{2}}. \end{aligned} \tag{17}$$

Thus, the 32 even elements are divided into two commuting sets composed of 16 elements each, corresponding to sectors of given $I_3 = \pm\frac{1}{2}$.

3.1. $U(1)\otimes SU(3)$ generators

3.1.1. The $U(1)\otimes SU(3)$ structure. The standard $SU(3)$ generators F_b's ($b = 1,\ldots,8$) are built from H_{nk} as in equation (68) of [1], with explicit form of H_{nk}'s given in appendix A. When the projections of F_b's onto the $I_3 = \pm 1/2$ subspaces are defined as

$$F_a^\pm = F_aI_{\pm\frac{1}{2}}, \tag{18}$$

the two sets of F_a^τ (with $\tau = \pm$ labelling the subspaces of definite $I_3 = \tau\frac{1}{2}$) operate in disjoint subspaces:

$$F_b^\pm F_c^\mp = 0. \tag{19}$$

The F_a^τ's satisfy standard $SU(3)$ commutation relations

$$\left[F_a^\tau, F_b^\tau\right] = 2\mathrm{i}f_{abc}F_c^\tau, \tag{20}$$

with f_{abc} being the relevant structure constants (see [1]).

Since all F_b's commute with Y, it is also natural to consider the projections of F_b^τ onto the triplet and singlet subspaces. With

$$Y_{-1}F_b^\tau = F_b^\tau Y_{-1} = 0, \tag{21}$$

it follows that F_b^τ's are equal to their projections onto the triplet subspace:

$$F_b^\tau = Y_{+\frac{1}{3}}F_b^\tau Y_{+\frac{1}{3}}. \tag{22}$$

Together, the eight generators F_b of $SU(3)$ and the six 'genuine' $SU(4)$ shift operators H_{m0}, H_{0m} make 14 generators. The 15th generator of $SU(4)$ is proportional to the $U(1)$ generator R, and in the same normalization it is

$$F_{15} \equiv \frac{1}{\sqrt{6}}R = \sqrt{6}YI_3, \tag{23}$$

with $R = 2H_{kk} = \sigma_k\otimes\sigma_k\otimes\sigma_3$, and its projections $R^\pm = RI_{\pm\frac{1}{2}}$ commuting with F_b^τ:

$$\left[R^\pm, F_b^\tau\right] = 0. \tag{24}$$

The four projection operators from the first two lines of equation (15) are constructed from elements $R^\pm$ and $I_{\pm\frac{1}{2}}$.

If X may be represented as a product of a certain definite number of elements C_k and $C_l^\dagger$, its $U(1)$ properties are specified by $\nu(X)$ defined as

$$[R, X] = +2\nu(X)X. \tag{25}$$

The value of $\nu(X)$ gives the number of $C_k^\dagger$'s minus the number of C_l's present in the product in question. Thus, $\nu(F_b) = 0$.

3.1.2. Transformations of C_k and $C_k^\dagger$. Shift operators H_{kl} act on C_n and $C_n^\dagger$ as follows (for any k, l, n):

$$[H_{kl}, C_n] = -\delta_{ln}C_k, \qquad \left[H_{kl}, C_n^\dagger\right] = +\delta_{kn}C_l^\dagger. \tag{26}$$

In particular, under $SU(3)$ transformations, elements C_k transform as a triplet, while $C_k^\dagger$ as an antitriplet:

$$[F_a, C_k] = \lambda_{akl}C_l, \qquad \left[F_a, C_k^\dagger\right] = -\lambda_{akl}^* C_l^\dagger, \tag{27}$$

where λ_{akl} may be read off from

$$\begin{aligned}
&[F_1, C_2] = [F_3, C_1] = [F_4, C_3] = \sqrt{3}[F_8, C_1] = -C_1,\\
&[F_1, C_1] = -[F_3, C_2] = [F_6, C_3] = \sqrt{3}[F_8, C_2] = -C_2,\\
&[F_4, C_1] = [F_6, C_2] = -\tfrac{\sqrt{3}}{2}[F_8, C_3] = -C_3,\\
&[F_2, C_2] = [F_5, C_3] = -\mathrm{i}C_1,\\
&[F_2, C_1] = -[F_7, C_3] = +\mathrm{i}C_2,\\
&[F_5, C_1] = [F_7, C_2] = +\mathrm{i}C_3
\end{aligned} \tag{28}$$

(the remaining commutators are zero). For $C_k^\dagger$ (antitriplet) the relevant commutation relations are obtained by taking the Hermitian conjugate of the above. Under $U(1)$ we have

$$[R, C_k] = -2C_k, \qquad \left[R, C_k^\dagger\right] = +2C_k^\dagger, \tag{29}$$

i.e. $\nu(C_k) = -\nu\left(C_k^\dagger\right) = -1$. Under ordinary reflections $\mathcal{P} = \exp\left(-\mathrm{i}\frac{\pi}{2}R\right) = -\mathrm{i}B$, element $C_k(C_k^\dagger)$ changes sign:

$$\mathcal{P}C_k\mathcal{P}^{-1} = -C_k. \tag{30}$$

3.2. Genuine generators of SU(4)

The six additional shift operators H_{m0} and H_{0m} satisfy

$$H_{0m} = (H_{m0})^\dagger \tag{31}$$

and constitute 'genuine' SU(4) operators. Their explicit forms are given in appendix A.

Shift operators H_{m0}^τ and H_{0m}^τ act between subspaces with different eigenvalues of Y:

$$\begin{aligned}
H_{m0}^{+} &= Y_{-1}^{+}H_{m0}Y_{+\frac{1}{3}}^{+}, \qquad & H_{m0}^{-} &= Y_{+\frac{1}{3}}^{-}H_{m0}Y_{-1}^{-},\\
H_{0m}^{+} &= Y_{+\frac{1}{3}}^{+}H_{0m}Y_{-1}^{+}, \qquad & H_{0m}^{-} &= Y_{-1}^{-}H_{0m}Y_{+\frac{1}{3}}^{-},
\end{aligned} \tag{32}$$

i.e. they connect the triplet and singlet $SU(3)$ subspaces with each other, as also indicated by subscripts '$m0$' and '$0m$' of our notation. The six 'genuine' Hermitean generators of $SU(4)$ are built from H_{m0}^τ and H_{0m}^τ as

$$F_{+n}^\tau = H_{n0}^\tau + H_{0n}^\tau, \qquad F_{-n}^\tau = \mathrm{i}\left(H_{n0}^\tau - H_{0n}^\tau\right). \tag{33}$$

Elements F_{-n}^τ describe simultaneous rotations in $\mathbf{x}$ and $\mathbf{p}$ spaces in mutually opposite senses (counterparts to ordinary simultaneous rotations in likewise senses).

3.2.1. Transformation properties of H_{m0} and H_{0m}. The $U(1) \otimes SU(3)$ transformation properties of H_{m0} and H_{0m} are (for any k, l, m)

$$\left[H^\tau_{kl}, H^\tau_{m0}\right] = +\delta_{mk} H^\tau_{l0} - \delta_{kl} H^\tau_{m0}, \qquad \left[H^\tau_{kl}, H^\tau_{0m}\right] = -\delta_{ml} H^\tau_{0k} + \delta_{kl} H^\tau_{0m}. \tag{34}$$

From the point of view of $SU(3)$ (traceless generators, i.e. either $k \neq l$, or appropriate linear combinations of terms with $k = l$), the second term on the rhs above does not contribute. Thus, the above equation shows that the $SU(3)$ transformation properties of H^τ_{m0} coincide with those of $C^\dagger_m$ in equation (26), i.e. with antitriplet, while the H_{0m}'s transform like C_m, i.e. an $SU(3)$ triplet. In other words, we have

$$\left[F^\tau_a, H^\tau_{0k}\right] = \lambda_{akm} H^\tau_{0m}, \qquad \left[F^\tau_a, H^\tau_{k0}\right] = -\lambda^*_{akm} H^\tau_{m0}. \tag{35}$$

From equation (34) we further obtain that elements H^τ_{m0} and H^τ_{0m} transform under $U(1)$ like

$$\left[R^\tau, H^\tau_{m0}\right] = -4H^\tau_{m0}, \qquad \left[R^\tau, H^\tau_{0m}\right] = +4H^\tau_{0m}, \tag{36}$$

i.e. H^τ_{m0} transform like a simple product of two C_k's (and not like a single $C^\dagger_m$), with R eigenvalues of C_k's simply added, while H^τ_{0m} transform like a product of two $C^\dagger_k$'s (see equations (41)). Obviously

$$\nu\left(H^\tau_{0m}\right) = -\nu\left(H^\tau_{m0}\right) = +2. \tag{37}$$

For completeness, we also give the anticommutators (for any k, l, m):

$$\left\{H^\pm_{kl}, H^\pm_{m0}\right\} = \mp\delta_{km} H^\pm_{l0}, \qquad \left\{H^\pm_{kl}, H^\pm_{0m}\right\} = \mp\delta_{lm} H^\pm_{0k}, \tag{38}$$

and the products of genuine $SU(4)$ shift operators themselves:

$$\begin{aligned} H^\tau_{m0} H^\tau_{n0} &= H^\tau_{0m} H^\tau_{0n} = 0, \\ H_{0n} H_{m0} &= \tfrac{1}{4}\delta_{nm}(1 + R) - H^+_{nm}, \\ H_{m0} H_{0n} &= \tfrac{1}{4}\delta_{nm}(1 - R) + H^-_{nm}. \end{aligned} \tag{39}$$

From the latter formulae one gets

$$\left[H^\tau_{0n}, H^\tau_{m0}\right] = \tfrac{1}{2}\delta_{nm} R^\tau - H^\tau_{nm}, \qquad \left\{H^\tau_{0n}, H^\tau_{m0}\right\} = \tfrac{1}{2}\delta_{nm} I_{\tau\frac{1}{2}} - \tau H^\tau_{nm}. \tag{40}$$

3.2.2. Transformations of C_k and $C^\dagger_k$. Under the action of H_{m0} and H_{0m}, matrices C_n and $C^\dagger_n$ transform as follows:

$$\begin{aligned} [H_{m0}, C_n] &= 0, \qquad & [H_{0m}, C_n] &= -\epsilon_{mnj} C^\dagger_j, \\ \left[H_{0m}, C^\dagger_n\right] &= 0, \qquad & \left[H_{m0}, C^\dagger_n\right] &= +\epsilon_{mnj} C_j. \end{aligned} \tag{41}$$

This translates into

$$[F_{+n}, C_k] = -\epsilon_{nkl} C^\dagger_l, \qquad \left[F_{+n}, C^\dagger_k\right] = +\epsilon_{nkl} C_l \tag{42}$$

and

$$[F_{-k}, C_l] = +\mathrm{i}\epsilon_{klm} C^\dagger_m, \qquad \left[F_{-k}, C^\dagger_l\right] = +\mathrm{i}\epsilon_{klm} C_m. \tag{43}$$

To summarize, the 32 even elements of the Clifford algebra are composed of the unit element and the 15 generators of $SU(4)$, with each of these 16 elements multiplied by $I_{\pm\frac{1}{2}}$. These two sets commute with each other. Under the $SU(3)$ transformations each set decomposes into two singlets (i.e. projection operators), an octet, a triplet and an antitriplet. All elements stay invariant under ordinary reflection. The full decomposition is given in table 1. The first two columns of the table describe transformation properties under $U(1)$ (the value of ν) and $SU(3)$ (representation). The four rightmost columns specify the left and right eigenvalues Y_l, Y_r (and Q_l, Q_r) of Y (and Q), defined for Y as

$$YZ = Y_l Z \qquad ZY = Y_r Z, \tag{44}$$

with $Z = H^+_{m0}, H^+_{0m}, F^+_b, \ldots$, etc, and similarly for charge Q (equation (10)).

Table 1. $U(1) \otimes SU(3)$ classification of 32 even elements of Clifford algebra. In the four rightmost columns the relevant left and right eigenvalues of Y and Q are given.

$U(1)$	$SU(3)$		Y_l	Y_r	Q_l	Q_r
		Sector $I_3 = +\frac{1}{2}$				
−2	$\bar{3}$	H^+_{m0}	−1	$+\frac{1}{3}$	0	+2/3
+2	3	H^+_{0m}	$+\frac{1}{3}$	−1	+2/3	0
0	8	F^+_b	$+\frac{1}{3}$	$+\frac{1}{3}$	+2/3	+2/3
0	1	Y^+_{-1}	−1	−1	0	0
0	1	$Y^+_{+\frac{1}{3}}$	$+\frac{1}{3}$	$+\frac{1}{3}$	+2/3	+2/3
		Sector $I_3 = -\frac{1}{2}$				
−2	$\bar{3}$	H^-_{m0}	$+\frac{1}{3}$	−1	−1/3	−1
+2	3	H^-_{0m}	−1	$+\frac{1}{3}$	−1	−1/3
0	8	F^-_b	$+\frac{1}{3}$	$+\frac{1}{3}$	−1/3	−1/3
0	1	Y^-_{-1}	−1	−1	−1	−1
0	1	$Y^-_{+\frac{1}{3}}$	$+\frac{1}{3}$	$+\frac{1}{3}$	−1/3	−1/3

4. Odd elements of Clifford algebra

The even elements of Clifford algebra are diagonal in I_3. In order to discuss the odd elements, we define weak isospin raising and lowering operators I_+ and I_-:

$$I_+ = \sigma_0 \otimes \sigma_0 \otimes \frac{\sigma_1 + \mathrm{i}\sigma_2}{\sqrt{2}} \propto A_1A_2A_3 + \mathrm{i}B_1B_2B_3, \tag{45}$$

with $I_- = I_+^\dagger$. With elements $I_\pm$ being odd, the odd elements of Clifford algebra, i.e. sets (3) and (4) are then obtained via multiplication by $I_\pm$ of the (even) elements of the first two sets. Obviously, the odd elements are off-diagonal in I_3 and change sign under ordinary reflection. All odd elements may be obtained from products of an odd number (one or three) of C_k's and $C_l^\dagger$, with these products multiplied from left and right by the (even) projection operators corresponding to subspaces of definite Y and I_3.

4.1. $SU(3)$ *triplets and antitriplets*

We now project C_k (either from the left or from the right) onto subspaces of definite Y and I_3, and define W_k, V_k, U_k as follows:

$$\begin{aligned} W_k &= \mathrm{i}Y^+_{-1}C_k = \mathrm{i}C_kY^-_{+\frac{1}{3}}, \qquad & V_k &= \mathrm{i}Y^+_{+\frac{1}{3}}C_k = \mathrm{i}C_kY^-_{-1}, \\ U_k &= \mathrm{i}Y^-_{+\frac{1}{3}}C_k = \mathrm{i}C_kY^+_{+\frac{1}{3}}, \qquad & 0 &= Y^-_{-1}C_k = C_kY^+_{-1}. \end{aligned} \tag{46}$$

They satisfy

$$W_k + V_k + U_k = \mathrm{i}C_k. \tag{47}$$

Explicit expressions for U_k, V_k, W_k are given in appendix A.

Since F_a commute with I_3 and Y, it follows that U_k, V_k and W_k transform under $SU(3)$ just like C_k, i.e. they are $SU(3)$ triplets. Thus, we have

$$\begin{aligned} [F_a, V_k] &= F_a^+V_k = \lambda_{akl}V_l, \\ [F_a, W_k] &= -W_kF_a^- = \lambda_{akl}W_l, \\ [F_a, U_k] &= F_a^-U_k - U_kF_a^+ = \lambda_{akl}U_l. \end{aligned} \tag{48}$$

The Hermitean conjugates $U_k^\dagger$, $V_k^\dagger$ and $W_k^\dagger$ transform like $C_k^\dagger$ and are antitriplets. Similarly, since R commutes with Y and I_3, it follows that U_k, V_k and W_k transform under $U(1)$ just like C_k:

$$[R, U_k] = -2U_k, \qquad [R, V_k] = -2V_k, \qquad [R, W_k] = -2W_k, \tag{49}$$

i.e. the values of ν are still well defined:

$$\nu(U_k) = \nu(V_k) = \nu(W_k) = -1. \tag{50}$$

Multiplication rules for elements U_k, V_l, W_m and their Hermitean conjugates are given in appendix B. When expressed in terms of U_k, V_l and W_m, the $U(1) \otimes SU(3)$ generators in $I_3 = \pm\frac{1}{2}$ subspaces are

$$H_{kl}^+ = -\tfrac{1}{4}\left(V_k V_l^\dagger + W_k W_l^\dagger - U_l^\dagger U_k\right), \qquad H_{kl}^- = +\tfrac{1}{4}\left(V_l^\dagger V_k + W_l^\dagger W_k - U_k U_l^\dagger\right). \tag{51}$$

For the genuine $SU(4)$ shift operators, we have

$$\begin{aligned} H_{m0}^+ &= +\tfrac{1}{4}\epsilon_{mkl} W_k U_l, & H_{m0}^- &= +\tfrac{1}{4}\epsilon_{mkl} U_k V_l, \\ H_{0m}^+ &= -\tfrac{1}{4}\epsilon_{mkl} U_k^\dagger W_l^\dagger, & H_{0m}^- &= -\tfrac{1}{4}\epsilon_{mkl} V_k^\dagger U_l^\dagger. \end{aligned} \tag{52}$$

4.2. SU(3) singlets

The only nonzero products that one can form from C_k's are $C_k C_l$ (i.e. H_{m0}) and the totally antisymmetric product $C_1 C_2 C_3$. We now define

$$\epsilon_{mkn} G_0 = \tfrac{1}{2} C_m C_k C_n = \tfrac{1}{8}\{[C_m, C_k], C_n\}, \tag{53}$$

with the mixed product $\{[C_m, C_k], C_n\}$ satisfying

$$\{[C_k, C_n], C_m\} = \{[C_m, C_k], C_n\} = \{[C_n, C_m], C_k\}. \tag{54}$$

The explicit form of G_0 is given in appendix A.

Using equation (16) we may rewrite G_0 also as

$$G_0 = -\tfrac{1}{2}\{H_{\underline{k}0}, C_{\underline{k}}\} = \tfrac{1}{16}\epsilon_{mn\underline{k}}\{[C_m, C_n], C_{\underline{k}}\}. \tag{55}$$

Element G_0 is diagonal in Y:

$$G_0 = Y_{-1}^+ G_0 = G_0 Y_{-1}^-, \qquad 0 = Y_{-1}^- G_0 = G_0 Y_{-1}^+. \tag{56}$$

Thus, G_0 (and $G_0^\dagger$) require $Y = -1$ and correspond to leptons. With $Y_{-1}^\pm$ being projection operators onto the $SU(3)$ singlet subspace, it is obvious from (21) that G_0 is a $SU(3)$ singlet:

$$[F_b, G_0] = 0. \tag{57}$$

This is also seen from equation (55) which (when summed over k) contains a trace of the product of a triplet C_k and an antitriplet H_{k0}. Element G_0 transforms under $U(1)$ as (e.g. using equation (29)):

$$[R, G_0] = -6G_0, \tag{58}$$

i.e. $\nu(G_0) = -3$.

In terms of U_k, V_l and W_m, we have

$$G_0 = +\frac{\mathrm{i}}{12}\epsilon_{mkn} W_m U_k V_n, \qquad G_0^\dagger = +\frac{\mathrm{i}}{12}\epsilon_{mkn} V_m^\dagger U_k^\dagger W_n^\dagger, \tag{59}$$

i.e., G_0, $G_0^\dagger$ are proportional to weak isospin raising and lowering operators in lepton subspace (consult the left and right eigenvalues of I_3 for W_k and V_l, equation (46)).

4.3. $SU(3)$ sextets and antisextets

In analogy to equation (55), we now form elements $G_{\{kl\}}$ defined as (for any k, l)

$$G_{\{kl\}} = \tfrac{1}{4}(\{H_{0k}, C_l\} + \{H_{0l}, C_k\}). \tag{60}$$

Element $G_{\{kl\}}$ is built as a symmetric combination of a product of two triplets: C_l and H_{0k}, and, consequently, it is a sextet. Its h.c. element $G^\dagger_{\{kl\}}$ is an antisextet. Their explicit forms are given in appendix A.

Under $U(1) \otimes SU(3)$ the sextet transforms as

$$[H_{kl}, G_{\{mn\}}] = -\delta_{lm} G_{\{kn\}} - \delta_{ln} G_{\{km\}} + \delta_{kl} G_{\{mn\}}. \tag{61}$$

For the $SU(3)$ (i.e. traceless) generators the last term on the rhs above does not contribute. It contributes only when one evaluates the commutator of the sum H_{kk} with $G_{\{mn\}}$, which leads to the following behaviour of $G_{\{mn\}}$ under $U(1)$:

$$[R, G_{\{mn\}}] = +2G_{\{mn\}}, \tag{62}$$

i.e. $\nu(G_{\{mn\}}) = +1$.

Furthermore, for any k, l one has

$$G_{\{kl\}} = Y^+_{+\frac{1}{3}} G_{\{kl\}} = G_{\{kl\}} Y^-_{+\frac{1}{3}}, \qquad 0 = Y^-_{+\frac{1}{3}} G_{\{kl\}} = G_{\{kl\}} Y^+_{+\frac{1}{3}}, \tag{63}$$

and similarly for $G^\dagger_{\{kl\}}$. Thus, $G_{\{kl\}}$ is equal to its projection onto the $Y = +\frac{1}{3}$ (i.e. quark) subspace.

When $k = l$ equation (60) reduces to

$$G_n \equiv G_{\{\underline{nn}\}} = +\tfrac{1}{2}\{H_{0\underline{n}}, C_{\underline{n}}\} = \tfrac{1}{16}\epsilon_{mk\underline{n}}\{[C^\dagger_m, C^\dagger_k], C_{\underline{n}}\}, \tag{64}$$

which constitutes a counterpart of equation (55) with mixed product $\{[C^\dagger_m, C^\dagger_k], C_n\}$ replacing $\{[C_m, C_k], C_n\}$ and satisfying

$$\{[C^\dagger_m, C^\dagger_k], C_n\} = \{[C_n, C^\dagger_m], C^\dagger_k\} = \{[C^\dagger_k, C_n], C^\dagger_m\}. \tag{65}$$

Explicitly, we have

$$G_n = Y_{+\frac{1}{3},n} I_+, \qquad G^\dagger_n = Y_{+\frac{1}{3},n} I_-, \tag{66}$$

with $G_n(G^\dagger_n)$ containing projection operators onto colour subspace $\#n$. These are essentially weak isospin raising and lowering operators for quarks of a given colour.

In terms of U_k, V_l, W_m, the elements $G_{\{kl\}}$ and $G^\dagger_{\{kl\}}$ may be expressed in various ways (using relations (B.4)) with the simplest form being

$$G_{\{kl\}} = -\frac{\mathrm{i}}{8} U^\dagger_r (\epsilon_{krs} U_l + \epsilon_{lrs} U_k) U^\dagger_s, \qquad G^\dagger_{\{kl\}} = +\frac{\mathrm{i}}{8} U_s (\epsilon_{krs} U^\dagger_l + \epsilon_{lrs} U^\dagger_k) U_r. \tag{67}$$

To summarize, under the $SU(3)$ transformations the 16 elements proportional to $\sigma_1 + \mathrm{i}\sigma_2$ decompose into an antitriplet, two triplets, a sextet and a singlet. Similar decomposition holds for the h.c. elements proportional to $\sigma_1 - \mathrm{i}\sigma_2$. The full decomposition is given in table 2.

5. The concept of mass

5.1. From $U(1) \otimes SU(3)$ to $SO(3)$

The basic physical idea motivating our approach is that the familiar macroscopic notions of space, time, etc are emergent concepts, which do not exist at the 'true' quantum level in any form other than a very rudimentary one. This idea was pursued in various contexts by many. For instance, Penrose suggested spin as a precursor of the concept of direction in the

Table 2. $U(1)\otimes SU(3)$ classification of 32 odd elements of Clifford algebra. In the four rightmost columns the relevant left and right eigenvalues of Y and Q are given.

$U(1)$	$SU(3)$		Y_l	Y_r	Q_l	Q_r
		Sector $I_{3,l}=+\frac{1}{2}, I_{3,r}=-\frac{1}{2}$				
+1	$\bar{3}$	$U_k^\dagger$	$+\frac{1}{3}$	$+\frac{1}{3}$	+2/3	−1/3
−1	3	V_k	$+\frac{1}{3}$	−1	+2/3	−1
−1	3	W_k	−1	$+\frac{1}{3}$	0	−1/3
+1	6	$G_{\{kl\}}$	$+\frac{1}{3}$	$+\frac{1}{3}$	+2/3	−1/3
−3	1	G_0	−1	−1	0	−1
		Sector $I_{3,l}=-\frac{1}{2}, I_{3,r}=+\frac{1}{2}$				
−1	3	U_k	$+\frac{1}{3}$	$+\frac{1}{3}$	−1/3	+2/3
+1	$\bar{3}$	$V_k^\dagger$	−1	$+\frac{1}{3}$	−1	+2/3
+1	$\bar{3}$	$W_k^\dagger$	$+\frac{1}{3}$	−1	−1/3	0
−1	$\bar{6}$	$G_{\{kl\}}^\dagger$	$+\frac{1}{3}$	$+\frac{1}{3}$	−1/3	+2/3
+3	1	$G_0^\dagger$	−1	−1	−1	0

3D space [10]. Page and Wootters proposed that quantum correlations could give rise to the macroscopic concept of time [11]. The general idea was succinctly expressed by Wheeler as 'Day One—quantum principle, Day Two—geometry' [12]. It was also pointed out that causality and quantum prescriptions, when combined, suggest the existence (or emergence) of a preferred frame [13] and absolute simultaneity. I believe therefore that mixing the ordinary 3D space with time, characteristic of the standard form of special relativity, should not be used as a starting point to seek the underlying 'true' quantum level. Instead, with quantum mechanics living in phase space, it is the mixing of the 3D space of positions with the 3D space of momenta which should be more appropriate. In line with these ideas, the concept of 3D rotations in ordinary space—understood as same-size and same-sense rotations in momentum and position subspaces—is in our approach extended to 6D rotations in phase space. This requires introduction of a new physical constant of dimension (momentum/position). At the quantum level, with the Planck constant at our disposal, the mass scale is then set. This quantum level is therefore expected to contain not only spin, but also some quantum ideas about the concept of mass. I believe therefore that a part of the problem of mass quantization should find its resolution at the level of 6D rotations in the algebra underlying nonrelativistic phase space. In other words, I think that this algebra (or perhaps its appropriate generalization) should lead to a joint quantum treatment of both masses and spins. The approach of [1–4] and of this paper constitute but a first step in that direction. Speaking more generally, the idea of 'space quantization' (or rather 'finding the underlying quantum-level precursors of 3D space') is naturally replaced in the phase space approach by similarly understood 'phase-space quantization'. Thus, when 'space' is understood as 'phase-space', the problem of 'space quantization' and the problem of elementary particle mass spectrum (and of their quantum numbers) seem closely related. I am tempted to think of them as of conceptually one and the same problem.

While the main idea of [1–3] is concerned with rotations in phase space, the treatment of reflections, although explicit, is extremely simplified when compared to what the SM tells us about the quantum world of elementary particles. Yet, reflection at the quantum level has to be properly treated if the space (phase space) points are to emerge with their real-world properties. This puts a question mark with respect to the introduction of the gauge principle into our present scheme. One could perhaps satisfy oneself with a 'mixed' (i.e. classical-quantum) level of theory and introduce the gauge structure at least for the $U(1) \otimes SU(3)$ part. We should be then back at the field-theoretical SM level of description. I think, however, that such a procedure will be fully legitimate only when a better understanding of how to generate space (phase space) points is reached. Consequently, in this paper we do not address the issue of gauge invariance. The latter should appear in its full form only on 'Day Three'.

In the standard model the mass-generating prescription (i.e. Higgs mechanism) lies outside of the strict gauge structure of the theory. Gauge interactions tell us nothing about the masses of the fundamental particles. In fact, these masses have to be put into SM by hand. Thus, in the SM the problem of mass appears to be separate from gauge interactions. Consequently, in our approach we should also expect some separation between the $U(1) \otimes SU(3)$ group structure (related to phase-space symmetries) and the mass generating mechanism. The overall scheme is thus anticipated to contain the following two ingredients:

(1) The mass-independent part, i.e. the symmetry group $U(1) \otimes SU(3)$ (combined with $SU(2)$). It is related to the nonrelativistic rotational (and reflectional) symmetry generalized to $O(6)$, with the generalization assumed valid at each point of our 3D world. Consequently, it should survive the emergence of space (and time) points and constitutes a conjectured precursor of the SM gauge group.
(2) A separate mass-defining prescription. This prescription must be tied to the *standard* concept of mass. This means it has to be related to the choice of which three coordinates of the 6D phase space are to be considered as the standard momentum coordinates, or, in other words, how $U(1) \otimes SU(3)$ is embedded into $SU(4)$, and which three of the eight $SU(3)$ generators constitute the ordinary rotations of our 3D macroscopic world. At the quantum level, therefore, the relevant quantum-level prescription for mass should be related to spin (and its quantization). Despite the connection of internal symmetry to the space degrees of freedom, our approach should evade the conclusions of the Coleman–Mandula theorem [14]. Indeed, the latter applies only to those symmetries which show up explicitly in the *S*-matrix approach, i.e. for ordinary particles with their standard concept of mass. Our quarks, however, though intimately related to phase-space symmetries [1–3], should be absent at the *S*-matrix level (i.e. confined), as the non-standard properties of quark mass (section 5.3) suggest. Obviously, this indicates that the issue of quark confinement and the description of systems composed of quarks constitute important questions to be addressed in our approach.

The appearance of the standard concept of mass is here linked with the restriction of the precursor of the unbroken part of the SM gauge group to ordinary $O(3)$. This may be considered strange. Yet, it seems to follow naturally from the assumptions of our approach, which look very appealing. Thus, one feels forced to accept their consequences. In summary, the problem of mass quantization is thought to belong to the underlying quantum level. On the other hand, the standard description of electromagnetic and strong interactions is expected to be the emergent one, with the underlying $U(1) \otimes SU(3)$ symmetry group conjectured to survive (in the form of gauge group) the procedure of mass quantization.

5.2. Lepton mass term

As argued above, we must tie our scheme to the standard concept of mass. Now, lepton mass is certainly standard since it appears in a relation which connects lepton energy (kinetic energy in nonrelativistic case), momentum and mass. Consequently, we must start from the lepton mass term. In order to discuss it, we first note that in our Clifford algebra there are only four elements which are both diagonal in Y and have $Y = -1$, as appropriate for the leptons. These are the (even) projection operators $Y^{\pm}_{-1}$ and the (odd) elements G_0 and $G_0^\dagger$. The latter two terms are $SU(3)$ singlets and $SO(3)$ scalars and, being odd like A_k (which is related to momentum), they constitute natural candidates for the elements related to lepton mass and kinetic energy.

As we show below, the nonrelativistic expression relating kinetic energy, mass and momentum gets linearized in our algebra by the consideration of the following linearized forms:

$$\begin{aligned}\mathcal{L}_1 &= Y_{-1}\big(A_k p_k - m_1 G_0 + E G_0^\dagger\big),\\ \mathcal{L}_2 &= \big(A_k p_k - m_2 G_0 + E G_0^\dagger\big) Y_{-1}.\end{aligned} \tag{68}$$

(Obviously, we could consider similar forms with $G_0 \leftrightarrow G_0^\dagger$.) The Y_{-1} projection operator appears here because our goal is the treatment of leptons. The presence of Y_{-1} (which commutes with ordinary 3D rotations) is irrelevant for the G_0 and $G_0^\dagger$ terms since $Y_{-1}G_0 = G_0$, $Y_{-1}G_0^\dagger = G_0^\dagger$, etc, but it affects the terms containing A_k.

Elements G_0 and $G_0^\dagger$ are not invariant under reflection. This is not a feature expected for mass (energy) terms. Yet, a similar lack of reflection invariance was observed in a fully-fledged Galilean framework [15]. This might be thought of as an indication that in the nonrelativistic Clifford algebra approach the treatment of reflection is oversimplified, as argued in the previous subsection. On the other hand, this persistent appearance of the violation of parity invariance might also be considered an interesting feature of the nonrelativistic approach. In any case, whatever point of view concerning the treatment of reflections one adopts, all our conclusions regarding the rotational properties should stay unaffected.

Since $Y_{-1}A_kY_{-1} \propto Y_{-1}(C_k^\dagger - C_k)Y_{-1} = 0$ it follows that

$$\begin{aligned}\mathcal{L}_1\mathcal{L}_2 &= Y_{-1}\big(A_kA_np_kp_n - m_1 E G_0 G_0^\dagger - m_2 E G_0^\dagger G_0\big)Y_{-1}\\ &= Y_{-1}\big(A_kA_np_kp_n - 2m_1 E Y^+_{-1} - 2m_2 E Y^-_{-1}\big)Y_{-1}\\ &= Y^+_{-1}(\mathbf{p}^2 - 2m_1E) + Y^-_{-1}(\mathbf{p}^2 - 2m_2E)\end{aligned} \tag{69}$$

where we used equation (B.7), i.e.

$$(G_0)^2 = \big(G_0^\dagger\big)^2 = 0, \qquad G_0G_0^\dagger = 2Y^+_{-1}, \qquad G_0^\dagger G_0 = 2Y^-_{-1}. \tag{70}$$

Thus, nonrelativistic expressions for kinetic energies of massive leptons of a given third component of isospin are obtained. To sum up, in the language of the Clifford algebra of nonrelativistic phase space the lepton mass term corresponds to element G_0 (or $G_0^\dagger$).

5.3. *Quark mass term*

We now want to transform the lepton mass element into the quark mass element. In [2, 3] it was shown that the transformation from lepton to quark #2 and vice versa is obtained by choosing $\phi = +\pi/2$ in the genuine phase-space rotation operator

$$\mathcal{R}_{02,\pm}(\phi) = \mathrm{e}^{+\mathrm{i}\phi F_{\pm 2}}. \tag{71}$$

Since this lepton-to-quark transformation constitutes an important part of the present paper, we repeat this calculation for the relevant projection operators.

In order to study the action of $\mathcal{R}_{02,\pm} \equiv \mathcal{R}_{02,\pm}(+\pi/2)$, we first note that $(F_{\pm n})^3 = F_{\pm n}$. Therefore

$$\mathcal{R}_{02,\pm} = 1 + \mathrm{i}F_{\pm 2} - (F_{\pm 2})^2. \tag{72}$$

Then, we calculate

$$\begin{aligned} F_{\pm 2}\mathcal{Y}_k &= \mp \mathrm{i}(1-\delta_{2k})F_{\mp 2}B - \delta_{2k}F_{\pm 2}, \\ (F_{\pm 2})^2\mathcal{Y}_k &= \tfrac{1}{2}(1-\delta_{2k})(\mathcal{Y}-\mathcal{Y}_2) - \tfrac{1}{2}\delta_{2k}(1-\mathcal{Y}_2), \end{aligned} \tag{73}$$

and find that

$$\mathcal{R}_{02,\pm}\mathcal{Y}_1\mathcal{R}_{02,\pm}^{-1} = -\mathcal{Y}_3, \qquad \mathcal{R}_{02,\pm}\mathcal{Y}_2\mathcal{R}_{02,\pm}^{-1} = +\mathcal{Y}_2, \qquad \mathcal{R}_{02,\pm}\mathcal{Y}_3\mathcal{R}_{02,\pm}^{-1} = -\mathcal{Y}_1. \tag{74}$$

Thus, the projection operators transform as

$$\begin{aligned} \mathcal{R}_{02,\pm}Y_{+\frac{1}{3},1}\mathcal{R}_{02,\pm}^{-1} &= Y_{+\frac{1}{3},1}, & \mathcal{R}_{02,\pm}Y_{+\frac{1}{3},2}\mathcal{R}_{02,\pm}^{-1} &= Y_{-1}, \\ \mathcal{R}_{02,\pm}Y_{+\frac{1}{3},3}\mathcal{R}_{02,\pm}^{-1} &= Y_{+\frac{1}{3},3}, & \mathcal{R}_{02,\pm}Y_{-1}\mathcal{R}_{02,\pm}^{-1} &= Y_{+\frac{1}{3},2}, \end{aligned} \tag{75}$$

i.e. quark #2 and lepton are interchanged.

In order to know how lepton mass element G_0 transforms, we need to know the individual actions of $F_{\pm 2}$ and $(F_{\pm 2})^2$ on various odd elements of Clifford algebra. The relevant formulae are gathered in appendix B. Using equations (B.18) we get

$$\begin{aligned} \mathcal{R}_{02,\pm}G_0\mathcal{R}_{02,\pm}^{-1} &= \mp G_2, & \mathcal{R}_{02,\pm}G_1\mathcal{R}_{02,\pm}^{-1} &= +G_1, \\ \mathcal{R}_{02,\pm}G_2\mathcal{R}_{02,\pm}^{-1} &= \mp G_0, & \mathcal{R}_{02,\pm}G_3\mathcal{R}_{02,\pm}^{-1} &= +G_3. \end{aligned} \tag{76}$$

Thus, the lepton mass element G_0 is transformed by the $\mathcal{R}_{02,\mp}$-induced transformations into $\pm G_2$ (and vice versa). Consequently, element G_2, a member of $SU(3)$ sextet, should correspond to the mass term of quark #2. Just like in the case of the commutation relations for a quark of a given colour (see [2, 3]), this mass term is not rotationally invariant. We shall comment on this lack of rotational invariance somewhat later.

The difference between $\mathcal{R}_{02,-}$- and $\mathcal{R}_{02,+}$-induced transformations is a $U(1) \otimes SU(3)$ phase factor $\mathcal{F}_{\{13\}}$:

$$\mathcal{F}_{\{13\}}\mathcal{R}_{02,-} \equiv \exp\left(\mathrm{i}\frac{\pi}{2}F_{\{13\}}\right)\mathcal{R}_{02,-} = \left(1 + \mathrm{i}F_{\{13\}} - F_{\{13\}}^2\right)\mathcal{R}_{02,-} = \mathcal{R}_{02,+}, \tag{77}$$

where

$$F_{\{13\}} \equiv \frac{1}{2}F_3 - \frac{1}{2\sqrt{3}}F_8 + \frac{1}{3}R = H_{11} + H_{33}. \tag{78}$$

This $U(1) \otimes SU(3)$ factor keeps commutation relations invariant. Under $\mathcal{F}_{\{13\}}$-induced transformations, elements G_0 and G_2 change signs, while G_1 and G_3 stay invariant. The total reflection, i.e. in particular $G_0, G_k \to -G_0, -G_k (k = 1, 2, 3)$, is obtained through the consecutive action of $\mathcal{F}_{\{13\}}$-, $\mathcal{F}_{\{23\}}$- and $\mathcal{F}_{\{12\}}$-induced transformations.

5.4. $SO(3)$ scalars

Individual elements corresponding to masses of coloured quarks, i.e.

$$G_k = G_{\{\underline{kk}\}} \tag{79}$$

belong to the $SU(3)$ sextet. When summed over quark colours, we obtain the trace of the symmetric matrix $G_{\{kl\}}$, i.e.

$$G \equiv G_{\{kk\}} = \pm\mathcal{R}_{0k,\mp}G_0\mathcal{R}_{0k,\mp}^{-1}, \tag{80}$$

which is an $SO(3)$ scalar. Thus, when added, the three rotationally-noninvariant individual quark mass elements give an $SO(3)$-invariant overall mass element. This provides an example of quark conspiration, a conjecture originally put forward in [1–4]. Interestingly (especially for a Clifford algebra approach), the quark mass element appears to be a trace of a rank 2 *symmetric* tensor in our 3D world.

The products of the quark mass elements G and $G^\dagger$ are (using equations (B.8) and (B.12)):

$$G^2 = (G^\dagger)^2 = 0, \qquad GG^\dagger = G_n G_n^\dagger = 2Y^+_{+\frac{1}{3}}, \qquad G^\dagger G = G_n^\dagger G_n = 2Y^-_{+\frac{1}{3}}, \tag{81}$$

mirroring the behaviour of the products of G_0 and $G_0^\dagger$ (equation (70)), but in the $Y = +\frac{1}{3}$ subspace. In particular, the latter two expressions, just like $G_0 G_0^\dagger$ and $G_0^\dagger G_0$, are invariant under both rotations and reflections.

The odd element I_+, with which we started in equation (45), is a linear superposition of $SU(3)$ singlet (G_0) and sextet ($G_{\{\underline{k}\underline{k}\}}$) terms:

$$I_+ = G_0 + G, \tag{82}$$

with G_0 and G proportional to weak isospin raising (lowering) operators in lepton and quark subspaces, respectively. The explicit form of G is given in appendix A.

5.5. *Relation to phase space*

By direct calculation using equations (42) and (43), or with the help of appendix B and equation (47), we find

$$\tilde{C}_k = \mathcal{R}_{0\underline{n},-} C_k \mathcal{R}^{-1}_{0\underline{n},-} = \delta_{k\underline{n}} C_{\underline{n}} + \epsilon_{k\underline{n}m} C_m^\dagger, \tag{83}$$

$$\tilde{C}'_k = \mathcal{R}_{0\underline{n},+} C_k \mathcal{R}^{-1}_{0\underline{n},+} = \delta_{k\underline{n}} C_{\underline{n}} + \mathrm{i}\epsilon_{k\underline{n}m} C_m^\dagger. \tag{84}$$

Obviously, when acted upon by the appropriate $U(1) \otimes SU(3)$ transformation (as in equation (77)), $\tilde{C}_k$ goes over into $\tilde{C}'_k$.

5.5.1. *$\mathcal{R}_{0\underline{2},-}$-induced transformations.*

For the $\mathcal{R}_{0\underline{2},-}$ transformations one gets

$$\tilde{A}_k = A_2 \delta_{2k} + \epsilon_{2kn} A_n, \qquad \tilde{B}_k = B_2 \delta_{2k} - \epsilon_{2kn} B_n, \tag{85}$$

and similarly for the related transformations of momenta and positions:

$$\tilde{p}_k = p_2 \delta_{2k} + \epsilon_{2kn} p_n, \qquad \tilde{x}_k = x_2 \delta_{2k} - \epsilon_{2kn} x_n. \tag{86}$$

Then

$$\mathbf{p}^2 \to \tilde{\mathbf{p}}^2 = \mathbf{p}^2, \qquad \mathbf{x}^2 \to \tilde{\mathbf{x}}^2 = \mathbf{x}^2. \tag{87}$$

This transformation is somewhat similar to ordinary rotations in that it does not 'mix' the physical momentum and physical position spaces. The difference is that rotation in each subspace proceeds here in the sense opposite to that in the other subspace.

5.5.2. *$\mathcal{R}_{0\underline{2},+}$-induced transformations.*

For the $\mathcal{R}_{0\underline{2},+}$ transformations one has

$$\tilde{A}'_k = \delta_{2k} A_2 - \epsilon_{2kn} B_n, \qquad \tilde{B}'_k = \delta_{2k} B_2 - \epsilon_{2kn} A_n \tag{88}$$

and similarly for the related transformations of momenta and positions:

$$\tilde{p}'_k = \delta_{2k} p_2 - \epsilon_{2kn} x_n, \qquad \tilde{x}'_k = \delta_{2k} x_2 - \epsilon_{2kn} p_n. \tag{89}$$

Thus

$$\mathbf{p}^2 \to (\tilde{\mathbf{p}}')^2 = x_1^2 + p_2^2 + x_3^2, \qquad \mathbf{x}^2 \to (\tilde{\mathbf{x}}')^2 = p_1^2 + x_2^2 + p_3^2. \tag{90}$$

The above behaviour is different from the $\mathcal{R}_{02,-}$ case. In the $\mathcal{R}_{02,+}$ case it is only its double application that leads to the preservation of $\mathbf{p}^2$ and $\mathbf{x}^2$.

5.5.3. Permutations of phase-space variables. Connection between the above two genuine $SU(4)$ transformations is given by the $U(1) \otimes SU(3)$ phase factor of equation (77). This factor changes some position coordinates into momenta and vice versa while keeping commutation relations invariant. Thus, it violates the association between e.g. lepton mass and momentum. Consequently, it seems natural to suppose that, when talking about the concept of mass, the $U(1) \otimes SU(3)$ freedom provided by $\mathcal{F}_{\{13\}}$ must be restricted (as argued earlier), and the choice between $\mathcal{R}_{02,\pm}$ limited to one possibility. I think that the appropriate choice is provided by $\mathcal{R}_{02,+}$, which does not keep the distinction between physical momenta and positions and may be considered more 'primitive' than the $\mathcal{R}_{02,-}$. As discussed in [3], this choice corresponds to the third alternative (obtained from equation(89) via ordinary 3D rotation by $\pi/2$ around the second axis) of the following four choices for the 'generalized momenta' P_k and 'generalized positions' X_k:

$$\begin{bmatrix} P_1, P_2, P_3 \\ X_1, X_2, X_3 \end{bmatrix} = \begin{bmatrix} p_1, p_2, p_3 \\ x_1, x_2, x_3 \end{bmatrix}, \begin{bmatrix} p_1, x_2, x_3 \\ x_1, p_2, p_3 \end{bmatrix}, \begin{bmatrix} x_1, p_2, x_3 \\ p_1, x_2, p_3 \end{bmatrix}, \begin{bmatrix} x_1, x_2, p_3 \\ p_1, p_2, x_3 \end{bmatrix}, \tag{91}$$

which are possible when pairs of permutations $(x_k, x_l) \leftrightarrow (p_k, p_l)$ are admitted. Such pairs of permutations keep an odd number of p_k's in $\mathbf{P}$ and x_k's in $\mathbf{X}$, and never permit a complete interchange $\mathbf{P} \leftrightarrow \mathbf{X}$. Thus, some distinction between physical momentum and physical position is still preserved. When expressed in terms of P_k and X_l, the commutation relations for each choice are then identical:

$$[P_k, P_l] = 0, \qquad [X_k, X_l] = 0, \qquad [X_k, P_l] = \mathrm{i}\delta_{kl}. \tag{92}$$

Alternatively, one may replace i with $-$ i, which is equivalent to (92) after genuine reflections in phase space are performed (e.g. $\mathbf{P} \to \mathbf{P}$, $\mathbf{X} \to -\mathbf{X}$). The above commutation formulae are invariant under 'generalized rotations', appropriately and independently defined for each of the four alternatives of equation (91).

In my opinion, the emergence of a mass element which—for a quark of given colour—is not rotationally invariant (and is associated with rotationally noninvariant generalized momenta which involve components of position) constitutes an asset of the approach, and should be welcomed in view of the unobservability of free quarks. An object which does not satisfy the standard relation between energy, mass and momentum clearly cannot be seen as a 'free particle'. This string-like idea on the origin of quark unobservability may coexist with the standard description of strong interactions in terms of non-Abelian $SU(3)$ gauge theory, which is viewed here as an emergent theory, tested at small distances only (i.e. at large momenta when standard quark masses should be neglected).

In fact, the application to quarks of the standard concept of mass leads to conceptual difficulties, discussed at some length in [4]. These difficulties are related to the use of *free* solutions of the Dirac equation (i.e. by setting $\not{p}\psi = m\psi$) for *confined* quarks, a standard procedure used e.g. when quark 'masses' are 'extracted' from the hadron-level data[3]. In fact, precisely such a use of *free* quark solutions for *confined* quarks has led to predictions for weak radiative hyperon decays [16] which disagreed with experiment in a dramatic way. The problem with the standard approach to quark mass is therefore not only conceptual. So far, the case of weak radiative hyperon decays has been solved only by a departure from the use of the Dirac equation for quarks [17], and a treatment of quarks at the level of current algebra symmetries [18, 19], with the concept of quark mass simply avoided.

[3] It is conceptually consistent to consider quark mass as a parameter introduced in a standard (lepton-like) way into the underlying SM Lagrangian, and to determine this parameter from the observed hadron masses after the latter are obtained from a *full nonperturbative QCD calculation*. The problem is that this is *not* the way in which quark masses (listed e.g. by Particle Data Group) are extracted from the data.

6. Conclusions and outlook

In this paper the Clifford algebra of nonrelativistic phase space is discussed in some detail. We have identified the element which may be associated with lepton mass and transformed it to the quark sector. The resulting individual quark mass element appears to be rotationally noninvariant. The total quark mass term was then obtained as a sum of three individual quark mass elements. In this term the individual rotationally-noninvariant contributions from three coloured quarks are combined in a rotationally invariant way, in line with the expectations that quarks should conspire to yield rotationally covariant structures.

We have discussed the one-particle system only. A further development of our approach should be presumably concerned with the description of nonrelativistic composite systems built of quarks. Hopefully, the idea of quark conspiracy could then be shown to work for such systems as well. The question of a connection between total spin and mass should also be addressed.

The issues related to the emergence of a continuum (space, time, etc), its symmetries, and in particular an extension of the whole scheme to link it to special relativity and accommodate gauge interactions, are clearly very important. In my opinion, however, they should be addressed only if a successful nonrelativistic treatment of composite systems and higher $SO(3)$ representations is developed.

Appendix A. Explicit expressions

A.1. Even elements

When summed over $I_3 = \pm\frac{1}{2}$, the even elements have the following explicit forms:

$$\begin{aligned}
H_{nk} &= \frac{1}{4}(\sigma_n \otimes \sigma_k + \sigma_k \otimes \sigma_n) \otimes \sigma_3 - \frac{\mathrm{i}}{4}\epsilon_{nkm}(\sigma_m \otimes \sigma_0 + \sigma_0 \otimes \sigma_m) \otimes \sigma_0, \\
H_{n0} &= -\frac{\mathrm{i}}{4}\left[(\sigma_0 \otimes \sigma_n - \sigma_n \otimes \sigma_0) \otimes \sigma_0 + \mathrm{i}\epsilon_{nkl}\sigma_k \otimes \sigma_l \otimes \sigma_3\right], \\
H_{0n} &= +\frac{\mathrm{i}}{4}\left[(\sigma_0 \otimes \sigma_n - \sigma_n \otimes \sigma_0) \otimes \sigma_0 - \mathrm{i}\epsilon_{nkl}\sigma_k \otimes \sigma_l \otimes \sigma_3\right], \\
1 &= \sigma_0 \otimes \sigma_0 \otimes \sigma_0.
\end{aligned} \tag{A.1}$$

The projection operators correspond to the specific combinations:

$$\begin{aligned}
Y^{\pm}_{-1} &= \frac{1}{4}(1 - \sigma_m \otimes \sigma_m) \otimes \frac{\sigma_0 \pm \sigma_3}{2}, \\
Y^{\pm}_{+\frac{1}{3}} &= \frac{1}{4}(3 + \sigma_m \otimes \sigma_m) \otimes \frac{\sigma_0 \pm \sigma_3}{2}.
\end{aligned} \tag{A.2}$$

A.2. Odd elements

The 16 elements proportional to $\sigma_1 + \mathrm{i}\sigma_2$ are (in the order: $SU(3)$ singlet, $SU(3)$ antitriplet, two $SU(3)$ triplets, $SU(3)$ sextet)

$$\begin{aligned}
G_0 &= \frac{1-y}{4} \otimes \frac{\sigma_1 + \mathrm{i}\sigma_2}{\sqrt{2}}, \\
U_k^\dagger &= -\frac{1}{2}(\sigma_0 \otimes \sigma_k + \sigma_k \otimes \sigma_0) \otimes \frac{\sigma_1 + \mathrm{i}\sigma_2}{\sqrt{2}}, \\
V_k &= \frac{1}{4}[(\sigma_0 \otimes \sigma_k - \sigma_k \otimes \sigma_0) - \mathrm{i}\epsilon_{kmn}\sigma_m \otimes \sigma_n] \otimes \frac{\sigma_1 + \mathrm{i}\sigma_2}{\sqrt{2}},
\end{aligned}$$

$$
\begin{aligned}
W_k &= \frac{1}{4}[(\sigma_0 \otimes \sigma_k - \sigma_k \otimes \sigma_0) + \mathrm{i}\epsilon_{kmn}\sigma_m \otimes \sigma_n] \otimes \frac{\sigma_1 + \mathrm{i}\sigma_2}{\sqrt{2}}, \\
G_{\{kl\}} &= \frac{1}{4}\,[\delta_{kl}(\sigma_0 \otimes \sigma_0 + \sigma_m \otimes \sigma_m) - (\sigma_k \otimes \sigma_l + \sigma_l \otimes \sigma_k)] \otimes \frac{\sigma_1 + \mathrm{i}\sigma_2}{\sqrt{2}}.
\end{aligned} \tag{A.3}
$$

The diagonal elements $G_{\{\underline{kk}\}}$ sum up to

$$
G = G_{\{kk\}} = \sum_k G_k = \frac{3+y}{4} \otimes \frac{\sigma_1 + \mathrm{i}\sigma_2}{\sqrt{2}}, \tag{A.4}
$$

with

$$
G_k = \frac{1}{4}(1 + y - 2y_k) \otimes \frac{\sigma_1 + \mathrm{i}\sigma_2}{\sqrt{2}}. \tag{A.5}
$$

Analogous expressions hold for 16 Hermitian-conjugated elements proportional to $\sigma_1 - \mathrm{i}\sigma_2$.

Appendix B. Products of elements

B.1. Odd–odd

B.1.1. Triplet–triplet. One finds

$$
W_k W_l = V_k V_l = U_k U_l = V_k W_l = W_k V_l = V_k U_l = U_k W_l = 0, \tag{B.1}
$$

and

$$
\begin{aligned}
W_k U_l &= 2\epsilon_{mkl} Y^+_{-1} H_{m0} Y^+_{+\frac{1}{3}} = 2\epsilon_{mkl} H^+_{m0}, \\
U_k V_l &= 2\epsilon_{mkl} Y^-_{+\frac{1}{3}} H_{m0} Y^-_{-1} = 2\epsilon_{mkl} H^-_{m0},
\end{aligned} \tag{B.2}
$$

with analogous formulae for the h.c. expressions. Furthermore

$$
W_k V_l^\dagger = W_k^\dagger V_l = V_k U_l^\dagger = V_k^\dagger U_l = U_k W_l^\dagger = U_k^\dagger W_l = 0, \tag{B.3}
$$

together with h.c. relations. Using equations (46) we have

$$
\begin{aligned}
W_k W_l^\dagger &= Y^+_{-1}\{C_k, C_l^\dagger\} Y^+_{-1} = 2\delta_{kl} Y^+_{-1}, \\
V_k^\dagger V_l &= Y^-_{-1}\{C_k^\dagger, C_l\} Y^-_{-1} = 2\delta_{kl} Y^-_{-1}, \\
V_k V_l^\dagger + U_l^\dagger U_k &= Y^+_{+\frac{1}{3}}\{C_k, C_l^\dagger\} Y^+_{+\frac{1}{3}} = 2\delta_{kl} Y^+_{+\frac{1}{3}}, \\
W_k^\dagger W_l + U_l U_k^\dagger &= Y^-_{+\frac{1}{3}}\{C_k^\dagger, C_l\} Y^-_{+\frac{1}{3}} = 2\delta_{kl} Y^-_{+\frac{1}{3}}, \\
V_k V_l^\dagger - U_l^\dagger U_k &= Y^+_{+\frac{1}{3}}[C_k, C_l^\dagger] Y^+_{+\frac{1}{3}} = -4H^+_{kl} - 2\delta_{kl} Y^+_{-1}, \\
U_k U_l^\dagger - W_l^\dagger W_k &= Y^-_{+\frac{1}{3}}[C_k, C_l^\dagger] Y^-_{+\frac{1}{3}} = -4H^-_{kl} + 2\delta_{kl} Y^-_{-1}.
\end{aligned} \tag{B.4}
$$

B.1.2. Triplet–singlet. One finds

$$
\begin{aligned}
&W_k G_0 = V_k G_0 = U_k G_0 = C_k G_0 = 0 = G_0 C_k = G_0 W_k = G_0 V_k = G_0 U_k, \\
&V_k^\dagger G_0 = U_k^\dagger G_0 = 0 = G_0 W_k^\dagger = G_0 U_k^\dagger
\end{aligned} \tag{B.5}
$$

and

$$
\begin{aligned}
W_k^\dagger G_0 &= -\mathrm{i} C_k^\dagger G_0 = 2\mathrm{i} Y^-_{+\frac{1}{3}} H_{k0} = 2\mathrm{i} H_{k0} Y^-_{-1} = 2\mathrm{i} H^-_{k0}, \\
G_0 V_k^\dagger &= -\mathrm{i} G_0 C_k^\dagger = 2\mathrm{i} Y^+_{-1} H_{k0} = 2\mathrm{i} H_{k0} Y^+_{+\frac{1}{3}} = 2\mathrm{i} H^+_{k0},
\end{aligned} \tag{B.6}
$$

and similarly for the h.c. expressions.

B.1.3. Singlet–singlet

$$(G_0)^2 = \left(G_0^\dagger\right)^2 = 0, \qquad G_0 G_0^\dagger = 2Y_{-1}^+, \qquad G_0^\dagger G_0 = 2Y_{-1}^-. \tag{B.7}$$

B.1.4. Sextet–sextet. From the isospin structure of sextet elements one has (for any k, l, m, n)

$$G_{\{kl\}} G_{\{mn\}} = 0 = G_{\{kl\}}^\dagger G_{\{mn\}}^\dagger. \tag{B.8}$$

The products $G_{\{kl\}} G_{\{mn\}}^\dagger$ and $G_{\{kl\}}^\dagger G_{\{mn\}}$ belong to two separate (isospin) subspaces, with the general formulae (valid for any k, l, m, n) being:

$$\begin{aligned} G_{\{kl\}} G_{\{mn\}}^\dagger = \tfrac{1}{2}\Big\{&\left(\tfrac{1}{2} + H_{jj}\right)(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}) \\ &- (\delta_{km} H_{ln} + \delta_{kn} H_{lm} + \delta_{lm} H_{kn} + \delta_{ln} H_{km})\Big\} I_{+\frac{1}{2}} \end{aligned} \tag{B.9}$$

and

$$\begin{aligned} G_{\{kl\}}^\dagger G_{\{mn\}} = \tfrac{1}{2}\Big\{&\left(\tfrac{1}{2} - H_{jj}\right)(\delta_{mk}\delta_{nl} + \delta_{nk}\delta_{ml}) \\ &+ (\delta_{mk} H_{nl} + \delta_{nk} H_{ml} + \delta_{ml} H_{nk} + \delta_{nl} H_{mk})\Big\} I_{-\frac{1}{2}}. \end{aligned} \tag{B.10}$$

Partial cases of the above formulae are (for $k \neq l$!)

$$\begin{aligned} G_{\{\underline{kl}\}} G_{\{\underline{kl}\}}^\dagger &= G_{\{\underline{kl}\}} G_{\{\underline{lk}\}}^\dagger = \tfrac{1}{2}\left(Y_{+\frac{1}{3},k}^+ + Y_{+\frac{1}{3},l}^+\right), \\ G_{\{\underline{kl}\}}^\dagger G_{\{\underline{kl}\}} &= G_{\{\underline{kl}\}}^\dagger G_{\{\underline{lk}\}} = \tfrac{1}{2}\left(Y_{+\frac{1}{3},k}^- + Y_{+\frac{1}{3},l}^-\right), \end{aligned} \tag{B.11}$$

and for any k, m:

$$\begin{aligned} G_k G_m^\dagger &= G_{\{\underline{kk}\}} G_{\{\underline{mm}\}}^\dagger = 2\delta_{k\underline{m}} Y_{+\frac{1}{3},\underline{m}}^+, \\ G_k^\dagger G_m &= G_{\{\underline{kk}\}}^\dagger G_{\{\underline{mm}\}} = 2\delta_{k\underline{m}} Y_{+\frac{1}{3},\underline{m}}^-. \end{aligned} \tag{B.12}$$

Furthermore, two more typical cases are (for $l \neq k, n$ and $n \neq k$)

$$G_{\{k\underline{l}\}} G_{\{\underline{l}n\}}^\dagger = -\tfrac{1}{2} H_{kn}^+, \qquad G_{\{k\underline{l}\}}^\dagger G_{\{\underline{l}n\}} = +\tfrac{1}{2} H_{nk}^-, \tag{B.13}$$

and (for $k \neq n$)

$$\begin{aligned} G_{\{\underline{kk}\}} G_{\{\underline{k}n\}}^\dagger &= G_{\{\underline{n}k\}} G_{\{\underline{nn}\}}^\dagger = -H_{kn}^+, \\ G_{\{\underline{kk}\}}^\dagger G_{\{\underline{k}n\}} &= G_{\{\underline{n}k\}}^\dagger G_{\{\underline{nn}\}} = +H_{nk}^-. \end{aligned} \tag{B.14}$$

All other types of products yield zero. To summarize, all expressions on the rhs of the above equations contain only the unit operator and the generators of $U(1) \otimes SU(3)$ projected upon subspaces with $I_3 = \pm\frac{1}{2}$.

B.1.5. Sextet–triplet. Of all 24 products of $G_{\{kl\}}$ and $G_{\{kl\}}^\dagger$ with U_k, V_k, W_k, $U_k^\dagger$, $V_k^\dagger$ and $W_k^\dagger$, only eight are nonzero, i.e.

$$\begin{aligned} G_{\{kl\}} U_n &= +\mathrm{i}\left(\epsilon_{mln} H_{km}^+ + \epsilon_{mkn} H_{lm}^+\right), \\ G_{\{kl\}}^\dagger U_n^\dagger &= -\mathrm{i}\left(\epsilon_{mln} H_{mk}^- + \epsilon_{mkn} H_{ml}^-\right), \\ G_{\{kl\}} W_n^\dagger &= -\mathrm{i}\left(\delta_{kn} H_{0l}^+ + \delta_{ln} H_{0k}^+\right), \\ G_{\{kl\}}^\dagger V_n &= +\mathrm{i}\left(\delta_{kn} H_{l0}^- + \delta_{ln} H_{k0}^-\right), \end{aligned} \tag{B.15}$$

and their h.c. versions.

B.1.6. Sextet–singlet. Here all products are zero because sextet and singlet elements correspond to different values of hypercharge.

B.2. Even–odd

B.2.1. Shift operators of $U(1) \otimes SU(3)$ and odd elements. The nonzero products of the $U(1) \otimes SU(3)$ shift operators with the odd elements are

$$\begin{aligned}
H^+_{kl}U^\dagger_n &= +\frac{1}{2}\delta_{kn}U^\dagger_l - \mathrm{i}\epsilon_{mnl}G_{\{mk\}},\\
H^+_{kl}V_n &= +\frac{1}{2}\delta_{kl}V_n - \delta_{nl}V_k,\\
H^+_{kl}W_n &= -\frac{1}{2}\delta_{kl}W_n,\\
H^+_{kl}G_0 &= -\frac{1}{2}\delta_{kl}G_0,\\
H^+_{kl}G_{\{mn\}} &= -\frac{\mathrm{i}}{4}(\delta_{ml}\epsilon_{nkr} + \delta_{nl}\epsilon_{mkr})U^\dagger_r\\
&\quad - \frac{1}{2}(\delta_{ml}G_{\{nk\}} + \delta_{nl}G_{\{mk\}} - \delta_{kl}G_{\{mn\}})
\end{aligned} \tag{B.16}$$

and

$$\begin{aligned}
H^-_{kl}U_n &= -\frac{1}{2}\delta_{ln}U_k + \mathrm{i}\epsilon_{mnk}G^\dagger_{\{ml\}},\\
H^-_{kl}V^\dagger_n &= +\frac{1}{2}\delta_{kl}V^\dagger_n,\\
H^-_{kl}W^\dagger_n &= -\frac{1}{2}\delta_{kl}W^\dagger_n + \delta_{kn}W^\dagger_l,\\
H^-_{kl}G^\dagger_0 &= +\frac{1}{2}\delta_{kl}G^\dagger_0,\\
H^-_{kl}G^\dagger_{\{mn\}} &= +\frac{\mathrm{i}}{4}(\delta_{mk}\epsilon_{nlr} + \delta_{nk}\epsilon_{mlr})U_r\\
&\quad + \frac{1}{2}\big(\delta_{mk}G^\dagger_{\{nl\}} + \delta_{nk}G^\dagger_{\{ml\}} - \delta_{kl}G^\dagger_{\{mn\}}\big).
\end{aligned} \tag{B.17}$$

B.2.2. Genuine generators of $SU(4)$ and odd elements. The nonzero products of H_{n0}, H_{0n} and $F_{\pm n}$ with odd elements of Clifford algebra are

$$\begin{aligned}
&F_{-n}V_k = +\mathrm{i}F_{+n}V_k = +\mathrm{i}H_{n0}V_k = \delta_{nk}G_0,\\
&F_{-n}V^\dagger_k = +\mathrm{i}F_{+n}V^\dagger_k = +\mathrm{i}H_{n0}V^\dagger_k = -\frac{\mathrm{i}}{2}\epsilon_{nkl}U_l + G^\dagger_{\{nk\}},\\
&F_{-n}W_k = -\mathrm{i}F_{+n}W_k = -\mathrm{i}H_{0n}W_k = -\frac{\mathrm{i}}{2}\epsilon_{nkl}U^\dagger_l + G_{\{nk\}},\\
&F_{-n}W^\dagger_k = -\mathrm{i}F_{+n}W^\dagger_k = -\mathrm{i}H_{0n}W^\dagger_k = \delta_{nk}G^\dagger_0,\\
&F_{-n}U_k = -\mathrm{i}F_{+n}U_k = -\mathrm{i}H_{0n}U_k = -\mathrm{i}\epsilon_{nkl}V^\dagger_l,\\
&F_{-n}U^\dagger_k = +\mathrm{i}F_{+n}U^\dagger_k = +\mathrm{i}H_{n0}U^\dagger_k = -\mathrm{i}\epsilon_{nkl}W_l,\\
&F_{-n}G_0 = -\mathrm{i}F_{+n}G_0 = -\mathrm{i}H_{0n}G_0 = V_n,\\
&F_{-n}G^\dagger_0 = +\mathrm{i}F_{+n}G^\dagger_0 = +\mathrm{i}H_{n0}G^\dagger_0 = W^\dagger_n,\\
&F_{-n}G_{\{kl\}} = +\mathrm{i}F_{+n}G_{\{kl\}} = +\mathrm{i}H_{n0}G_{\{kl\}} = \frac{1}{2}\delta_{nk}W_l + \frac{1}{2}\delta_{nl}W_k,
\end{aligned}$$

$$F_{-n}G^{\dagger}_{\{kl\}} = -\mathrm{i}F_{+n}G^{\dagger}_{\{kl\}} = -\mathrm{i}H_{0n}G^{\dagger}_{\{kl\}} = \frac{1}{2}\delta_{nk}V^{\dagger}_l + \frac{1}{2}\delta_{nl}V^{\dagger}_k. \qquad \text{(B.18)}$$

For completeness, we also specify how U_k, V_k and W_k transform under $\mathcal{R}_{02,\pm}$-induced transformations:

$$\begin{aligned} \mathcal{R}_{02,-}V_k\mathcal{R}^{-1}_{02,-} &= \delta_{2k}W_2 + \tfrac{1}{2}\epsilon_{2kl}U^{\dagger}_l - \mathrm{i}(1-\delta_{2k})G_{\{2k\}}, \\ \mathcal{R}_{02,-}W_k\mathcal{R}^{-1}_{02,-} &= \delta_{2k}V_2 + \tfrac{1}{2}\epsilon_{2kl}U^{\dagger}_l + \mathrm{i}(1-\delta_{2k})G_{\{2k\}}, \\ \mathcal{R}_{02,-}U_k\mathcal{R}^{-1}_{02,-} &= \delta_{2k}U_2 + \epsilon_{2kl}\left(V^{\dagger}_l + W^{\dagger}_l\right) \end{aligned} \qquad \text{(B.19)}$$

and

$$\begin{aligned} \mathcal{R}_{02,+}V_k\mathcal{R}^{-1}_{02,+} &= \delta_{k2}W_2 + \frac{\mathrm{i}}{2}\epsilon_{2kl}U^{\dagger}_l + (1-\delta_{k2})G_{\{2k\}}, \\ \mathcal{R}_{02,+}W_k\mathcal{R}^{-1}_{02,+} &= \delta_{k2}V_2 + \frac{\mathrm{i}}{2}\epsilon_{2kl}U^{\dagger}_l - (1-\delta_{k2})G_{\{2k\}}, \\ \mathcal{R}_{02,+}U_k\mathcal{R}^{-1}_{02,+} &= \delta_{k2}U_2 + \mathrm{i}\epsilon_{2kl}\left(V^{\dagger}_l + W^{\dagger}_l\right), \end{aligned} \qquad \text{(B.20)}$$

with analogous equations for the Hermitean conjugates.

Under $\mathcal{R}_{02,\pm}$-induced transformations, the off-diagonal ($k \neq l$) elements $G_{\{kl\}}$ transform as

$$\begin{aligned} \mathcal{R}_{02,-}G_{\{12\}}\mathcal{R}^{-1}_{02,-} &= +\mathrm{i}\mathcal{R}_{02,+}G_{\{12\}}\mathcal{R}^{-1}_{02,+} = +\frac{\mathrm{i}}{2}(W_1 - V_1), \\ \mathcal{R}_{02,-}G_{\{13\}}\mathcal{R}^{-1}_{02,-} &= \mathcal{R}_{02,+}G_{\{13\}}\mathcal{R}^{-1}_{02,+} = G_{\{13\}}, \\ \mathcal{R}_{02,-}G_{\{23\}}\mathcal{R}^{-1}_{02,-} &= +\mathrm{i}\mathcal{R}_{02,+}G_{\{23\}}\mathcal{R}^{-1}_{02,+} = +\frac{\mathrm{i}}{2}(W_3 - V_3). \end{aligned} \qquad \text{(B.21)}$$

The corresponding formulae for $G_{\{\underline{kk}\}}$ are given in equation (76).

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